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General Framework for Dynamic Substructuring: History, Review, and Classification of Techniques

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Four decades after the development of the first dynamic substructuring techniques, there is a necessity to classify the different methods in a general framework that outlines the relations between them. In this paper, a certain vision on substructuring methods is proposed, by recalling important historical milestones that allow us to understand substructuring as a domain decomposition concept. Thereafter, based on the dual and primal assembly of substructures, a general framework for the classification of the methods is presented. This framework allows us to indicate how the various classes of methods, proposed along the years, can be derived from a clear mathematical description of substructured problems. Current bottlenecks in experimental dynamic substructuring, as well as solutions found in literature, will also be briefly discussed.

Nomenclature

$\mathbf{B}$	=	signed Boolean matrix
$\mathbf{C}$	=	damping matrix
$\mathbf{f}$	=	vector of external forces
$\mathbf{G}$	=	matrix of associated modes
$\mathbf{g}$	=	vector of connecting forces
$\mathbf{K}$	=	stiffness matrix
$\mathbf{L}$	=	Boolean localization matrix
$\mathbf{M}$	=	mass matrix
$\mathbf{q}$	=	vector of unique degrees of freedom
$\mathbf{R}$	=	reduction matrix
$\mathbf{r}$	=	vector of residual forces
t	=	time
$\mathbf{u}$	=	vector of degrees of freedom
$\mathbf{Y}$	=	receptance matrix
$\mathbf{Z}$	=	dynamic stiffness matrix
$\boldsymbol{\eta}$	=	vector of generalized coordinates
$\boldsymbol{\lambda}$	=	vector of Lagrange multipliers
$\boldsymbol{\xi}$	=	vector of unique generalized coordinates
ω	=	circular frequency
$\star_m$	=	pertaining to a modal domain property
$\star^{(s)}$	=	pertaining to a structure named s
$\tilde{\star}$	=	primal assembled matrix

I. Introduction

DYNAMIC substructuring (DS) has played a significant role in the field of structural dynamics and continues to be of great value. Performing the analysis of a structural system componentwise has some important advantages over global methods where the entire problem is handled at once:

1) It allows evaluating the dynamic behavior of structures that are too large or complex to be analyzed as a whole. For experimental analysis, this is true for large and complex systems such as aircraft. For numerical models, this holds when the number of degrees of freedom is such that solution techniques cannot find results in a reasonable time.

2) By analyzing the subsystems, local dynamic behavior can be recognized more easily than when the entire system is analyzed. Thereby, DS allows identification of local problems as well as efficient local optimization. Also, dynamic substructuring allows the elimination of local subsystem behavior which has no significant impact on the assembled system. This results in a simple representation of the component's dynamics (e.g., an effective mass criteria) and, consequently, in an additional reduction of analysis time.

3) Dynamic substructuring gives the possibility of combining modeled parts (discretized or analytical) and experimentally identifying components.

4) It allows sharing and combining substructures from different project groups.

The goal of this paper is to present a general framework which allows for classification of dynamic substructuring methods and highlights the interrelations and differences between the many variants published. It is indeed peculiar that, despite the fact that dynamic substructuring concepts have been used and investigated for many years, such general overviews on the subject have only rarely been proposed. Starting with a historical overview in Sec. II, the general framework will be presented in Sec. III. According to this framework, Sec. IV presents a classification of the different DS techniques and further details different strategies existing for frequency-based substructuring and for component-mode synthesis. The paper is concluded with a discussion on some of the major challenges in DS that require future research, namely the bottlenecks associated with the coupling of experimentally obtained substructures.

II. Random Walk in History

In this section, substructuring techniques are put in perspective by recalling some important historical milestones. Certainly, we do not claim to propose an exhaustive or even an objective historical review. Instead, a certain amount of subjectivity will be used to place important contributions to substructuring on the timeline, to describe often forgotten links between them.

Historically, the roots of substructuring concepts can be found in the field of *domain decomposition*. As discussed in this section, dynamic substructuring essentially can be seen as a special class of domain decomposition. The domain decomposition paradigm originates from the desire to analyze complex problems by considering separately the problem of its components and the problem of finding the interface solution. In other words, finding the solution to the local problems at subparts level, assuming the

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interface data are known. This corresponds to the old principle of "divide and conquer."

In 1890, Schwarz [1] imagined an iterative procedure based on domain decomposition to prove the existence and uniqueness of the solution of a Laplace problem in a domain made of a circle and a rectangle. The proof was based on the fact that the uniqueness of the solutions on a circle and a rectangle separately were known, and the iterative procedure between the subdomains could be shown to converge to a unique solution. The domain handled by Schwarz has become the icon of the domain decomposition community and will be used here to illustrate the discussion (see Fig. 1). More details on domain decomposition techniques can be found in the many reference books on the subject (e.g., [2,3]).

The publication of Schwarz [1] opened a new way to iteratively find solutions of analytical problems on complex domains. However, most of the analytical models of coupled continuous subdomains do not have a closed-form solution. To circumvent this problem, approximation and discretization techniques were developed (Fig. 1), such as the Rayleigh–Ritz approach [4], the boundary element technique, and the finite element method. In these methods, an approximate solution is sought by describing the infinite space of admissible solution functions as a combination of approximation functions. In the case of finite elements, the physical space is decomposed into subdomains described by piecewise defined approximation functions. The finite elements can therefore be seen as subdomains, which are based on simpler and approximate fields. In the view of this paper, these discretization methods, similar to the decomposition of the physical problem in the Schwarz method [1], can be considered as "first level" domain decomposition techniques, as indicated in Fig. 1.

Eventually, the finite element method emerged as a very efficient and versatile technique, and soon took a leading role [5,6]. With the invention of the microprocessor, system modeling quickly evolved and a large variety of physical problems could now be solved in a detailed and accurate way.

Driven by the desire to analyze ever larger and more complex problems, scientists and engineers searched for methods to optimize the efficiency of their (discretized) calculations. Various approaches were developed. The first steps to speed up the calculations consisted in replacing the direct solvers, basically variants of the factorization techniques of Gauss, by iterative solvers. A major milestone in iterative solvers was the conjugate gradient solver proposed in 1952 by Hestnes and Stiefel [7]. However, it soon became clear that iterative approaches were lacking robustness and could barely be used for complex engineering problems. Indeed, the bad mathematical conditioning of the problems, resulting from the discretization and the lack of smart preconditioners, induced a slow convergence. Iterative techniques therefore remained mainly an academic curiosity until the 1980s, when, sparked by the advent of parallel computing, iterative solvers became highly popular.

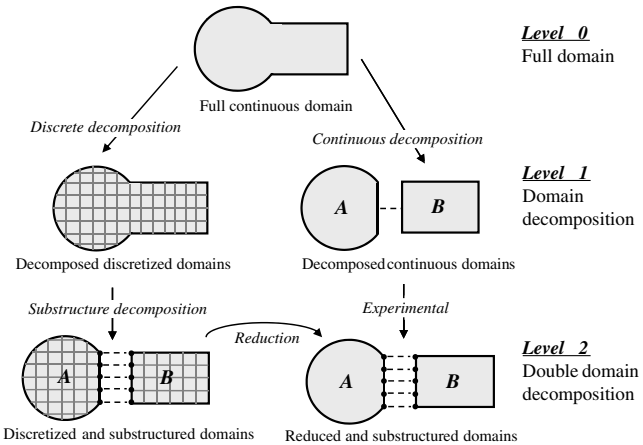

Fig. 1 Dynamic substructuring and its relation to domain decomposition.

Modern parallel computing techniques, used to solve complex mechanical engineering problems, are often based on the concept of domain decomposition. The problem is divided into subdomains that are handled by different processors, whereas the interface coupling problem is solved iteratively using the local solutions at the subdomain levels. This can be seen as a second level domain decomposition if one considers, as explained previously, that the discretization of the problem is a first decomposition level (see Fig. 1, left column).

For dynamics modeling, a way to further increase the analysis efficiency consists in reducing the complexity of the individual subdomains. After finding approximate local solutions, represented in a subspace of the physical degrees of freedom, the subdomains are coupled. In that case, the subdomains, called substructures in the dynamic context, are seen as components of the system represented by general responses and no longer through their detailed discretization. This is indicated by the "reduction" arrow in Fig. 1. This so-called dynamic substructuring method can also be obtained considering that the subparts are characterized by their experimental behavior. This can be seen as modifying the physical, substructured (but nondiscretized) model by replacing the mathematical description of the domains by experimentally obtained information. These methods are referred to as experimental substructuring (see vertical arrow in the right column of Fig. 1).

The first dynamic substructuring ideas were developed as reduction techniques and were probably triggered by the paper of Hurty in 1960 [8] and further worked out in [9]. In the same time period, a method using a branched vision of the organization of substructures was proposed by Gladwell [10]. These methods were soon known under the name "component-mode synthesis" (CMS), where the term "modes" includes all kinds of structural modes (e.g., exact eigenmodes, approximate modes, static modes, interface modes, etc.). Rapidly, the scientific and engineering communities discovered the benefits of dynamic substructuring and component-mode synthesis became an important research topic in the field of structural dynamics. Some major developments followed shortly, resulting in the classic methods by Craig and Bampton [11], Rubin [12], and MacNeal [13] in the late 1960s and 1970s.

In the 1980s, substructure coupling techniques became attractive to the experimental community, due to ever more accurate and faster testing equipment. However, the first attempts were already performed with experimentally obtained modal information in the early 1970s [14]. A decade later, coupling techniques were directly applied to measured frequency response functions (FRFs). At first, these methods dealt with structural dynamic modification (SDM) problems, with the aim to alter the dynamic behavior of a base structure by coupling a "modification" structure (usually lumped masses or springs). Although structural modification techniques are generally not considered as substructuring techniques, the two concepts are in fact identical, as observed in [15–17].

One of the first steps toward frequency-based coupling techniques was taken by Crowley et al., who proposed a structural modification method called SMURF (structural modification using experimental frequency response functions) [18]. However, this method failed to gain popularity with the broad public. A few years later, Jetmundsen et al. formulated the classical FRF-based substructuring (FBS) method [19], which was more efficient and more accurate than the impedance modeling method available at that time [20] (see Sec. IV.A for more details).

Summarizing, dynamic substructuring techniques can historically be placed in the framework of domain decomposition as a second level decomposition. The dynamic substructuring theory can be used for numerical and experimental data.

III. General Framework for Dynamic Substructuring

This section is similar to the outline in [21], Chap. 17, but a different perspective is proposed here, focusing on substructuring methods in general and not only in the context of model reduction. In the framework proposed here, the structural dynamics are therefore

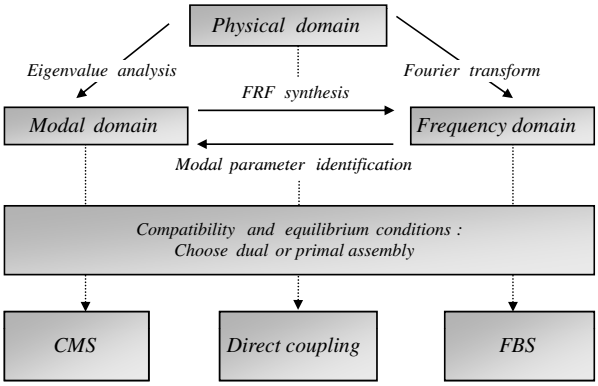

Fig. 2 Representation of system dynamics in three domains.

analyzed in three distinct domains: the physical, modal, and frequency domains.

In the physical domain, the structure is characterized by its mass, stiffness, and damping distributions, which are given by the corresponding stiffness, mass, and damping matrices for a discretized linear(ized) model. A structure in the frequency domain is seen through its frequency response functions. In the modal domain, the dynamic behavior of a structure is interpreted as a combination of modal responses: the system matrices are projected on the modal basis which, generally speaking, can be any basis representing the structural response. From a theoretical perspective, the same information is contained in all different representations (assuming no model reduction is performed). This is schematically shown in Fig. 2.

Substructures are structures that interact with their neighboring structures. When two or more substructures are to be coupled, two conditions must always be satisfied, regardless of the coupling method used:

1) Compatibility of the substructures' displacements at the interface is the so-called compatibility condition.

2) Force equilibrium on the substructures' interface degrees of freedom is called the equilibrium condition.

Knowledge of the dynamic models at substructure level and proper application of the coupling conditions allows one to calculate the response of the coupled system. Depending on whether one chooses displacement or forces as unknown at the interface, a primal or dual assembled system of equations is obtained, as shown schematically in Fig. 2. Note that, in some methods, both interface forces and displacements are chosen as interface unknowns, either separately (in the so-called three field formulations) or as a combination (Robin-type interface conditions, see, e.g., [22]). This will not be discussed here. Next, a general coupling framework will be presented in the three different domains in Secs. III.A–III.C.

A. Coupling in the Physical Domain

Let us consider the system as being described by its mass, damping, and stiffness matrices as obtained from the mechanical and geometrical properties of the system. We call that the physical domain. The equations of motion in the physical domain of a discrete/discretized dynamic subsystem s may be written as

$$\mathbf{M}^{(s)}\ddot{\mathbf{u}}^{(s)}(t) + \mathbf{C}\dot{\mathbf{u}}^{(s)}(t) + \mathbf{K}^{(s)}\mathbf{u}^{(s)}(t) = \mathbf{f}^{(s)}(t) + \mathbf{g}^{(s)}(t) \qquad (1)$$

Here $\mathbf{M}^{(s)}$, $\mathbf{C}^{(s)}$, and $\mathbf{K}^{(s)}$ are the mass, damping, and stiffness matrices of substructure s, $\mathbf{u}^{(s)}$ denotes its vector of degrees of freedom, $\mathbf{f}^{(s)}$ is the external force vector, and $\mathbf{g}^{(s)}$ is the vector of connecting forces with the other substructures. In this context, the connecting forces can be considered as constraining forces associated to the compatibility conditions. Note that, in these equations, it is implicitly assumed that the system is linear (the mass, damping, and stiffness properties are independent of the state of the system) and that it is time invariant (i.e., constant parameters). Although a similar framework could be written for nonlinear and time-variant systems, this will not be discussed here.

Note that whereas dynamic substructuring concepts of linear systems have been around for many years, dynamic substructuring of nonlinear systems is still in its infancy. Some methods have been proposed for the reduction and coupling [23,24] of nonlinear systems, but no "standard" method (like the Craig–Bampton method for linear systems [11]) has been established yet. Another way of handling nonlinear subsystems in a substructuring analysis is by applying real-time substructuring methods [25]. In these techniques, no explicit dynamic model of the nonlinear substructure has to be constructed, but the nonlinear dynamic behavior can be taken into account experimentally by including the substructure in the real-time substructuring loop.

The equations of motion of the n substructures that are to be coupled can be rewritten in a block-diagonal format as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{g} \qquad (2)$$

with

$$\mathbf{M} \triangleq \mathrm{diag}(\mathbf{M}^{(1)}, \ldots, \mathbf{M}^{(n)}) = \begin{bmatrix} \mathbf{M}^{(1)} & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & \mathbf{M}^{(n)} \end{bmatrix}$$

$$\mathbf{C} \triangleq \mathrm{diag}(\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(n)}) \qquad \mathbf{K} \triangleq \mathrm{diag}(\mathbf{K}^{(1)}, \ldots, \mathbf{K}^{(n)})$$

$$\mathbf{u} \triangleq \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(n)} \end{bmatrix}, \qquad \mathbf{f} \triangleq \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(n)} \end{bmatrix}, \qquad \mathbf{g} \triangleq \begin{bmatrix} \mathbf{g}^{(1)} \\ \vdots \\ \mathbf{g}^{(n)} \end{bmatrix}$$

For the sake of simplicity, the explicit time dependence has been omitted here. Next, the compatibility condition can be expressed by

$$\mathbf{B}\mathbf{u} = \mathbf{0} \qquad (3)$$

The $\mathbf{B}$ matrix operates on the interface degrees of freedom (DOF) and is a signed Boolean matrix if the interface degrees of freedom are matching (hence, for conforming discretizations on the interface). Note that sometimes the substructures do not originate from a partitioning of a global mesh: in some applications, the substructures are meshed independently or the interface meshes are sliding like in contact problems. In that case, the interface compatibility is usually enforced through nodal collocation or by using weak interface compatibility formulations, so that the compatibility condition can still be written as in Eq. (3) but now the operator $\mathbf{B}$ is no longer Boolean (see for instance [26]). Non-Boolean interface matrices arise also in other general multipoint constraints, such as when joint constraints are defined between components in multibody dynamics. The discussion in this paper is valid both when $\mathbf{B}$ is Boolean or not. If $\mathbf{B}$ is a signed Boolean matrix, the compatibility condition states that any pair of matching interface degrees of freedom $u^{(k)}$ and $u^{(l)}$ must have the same displacement, i.e., $u^{(k)} - u^{(l)} = 0$. More details on the formulation of the Boolean matrix $\mathbf{B}$ can be found in the Appendix.

The equilibrium condition is expressed by

$$\mathbf{L}^{T}\mathbf{g} = \mathbf{0} \qquad (4)$$

where the matrix $\mathbf{L}$ is the Boolean matrix localizing the interface DOF of the substructures in the global dual set of DOF. (Let us note that it is often stated that this equilibrium condition enforces the exact equilibrium on the interface between substructures, forgetting that the discretization process enforces equilibrium only in a weak sense and thus introduces an equilibrium error for the underlying continuous problem. Nevertheless, in this paper, we will call "exact" the solution and interface conditions related to the nonreduced discretized problem.) The expression states that when the dual connection forces are summed, their resultant must be equal to zero, i.e., $g^{(k)} + g^{(l)} = 0$. More details can be found in the Appendix. The total system is now described by Eqs. (2–4):

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{g} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \\ \mathbf{L}^T\mathbf{g} = \mathbf{0} \end{cases} \tag{5}$$

Note that Eq. (5) describes the coupling between *any* number of substructures with any number of arbitrary couplings. From this set of equations, the coupled system can be obtained in either a primal or a dual way, as discussed next.

1. Primal Formulation in Physical Domain

In a primal formulation, a unique set of interface degrees of freedom is defined, and the interface forces are eliminated as unknowns using the interface equilibrium. Classically, finite element models are assembled in this primal manner. Mathematically, this is obtained by stating that

$$\mathbf{u} = \mathbf{L}\mathbf{q} \tag{6}$$

where $\mathbf{q}$ is the unique set of interface DOF for the system and $\mathbf{L}$ is the Boolean matrix introduced earlier. Because Eq. (6) indicates that the substructure DOF are obtained from the unique set $\mathbf{q}$, it is obvious that the compatibility condition (3) is satisfied for any set $\mathbf{q}$, namely

$$\mathbf{B}\mathbf{u} = \mathbf{B}\mathbf{L}\mathbf{q} = \mathbf{0} \quad \forall\ \mathbf{q}$$

Hence, $\mathbf{L}$ actually represents the nullspace of $\mathbf{B}$ or vice versa:

$$\begin{cases} \mathbf{L} = \text{null}(\mathbf{B}) \\ \mathbf{B}^T = \text{null}(\mathbf{L}^T) \end{cases} \tag{7}$$

This is a very useful property when calculating the response of the coupled system because, in the assembly process, only one Boolean matrix needs to be formulated. The construction of these Boolean matrices, as well as an explicit computation of the nullspaces, is discussed in more detail in the Appendix.

Because the compatibility condition in Eq. (5) is satisfied by the choice of the unique set $\mathbf{q}$, the system is now described by

$$\begin{cases} \mathbf{M}\mathbf{L}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{L}\dot{\mathbf{q}} + \mathbf{K}\mathbf{L}\mathbf{q} = \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T\mathbf{g} = \mathbf{0} \end{cases}$$

Premultiplication of the equilibrium equations by $\mathbf{L}^T$ and noting that according to the equilibrium condition $\mathbf{L}^T\mathbf{g}$ is equal to zero, the primal assembled system reduces to

$$\tilde{\mathbf{M}}\,\ddot{\mathbf{q}} + \tilde{\mathbf{C}}\,\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \tilde{\mathbf{f}} \tag{8}$$

with the primal assembled system matrices defined by

$$\begin{cases} \tilde{\mathbf{M}} \triangleq \mathbf{L}^T\mathbf{M}\mathbf{L} \\ \tilde{\mathbf{C}} \triangleq \mathbf{L}^T\mathbf{C}\mathbf{L} \\ \tilde{\mathbf{K}} \triangleq \mathbf{L}^T\mathbf{K}\mathbf{L} \\ \tilde{\mathbf{f}} \triangleq \mathbf{L}^T\mathbf{f} \end{cases}$$

2. Dual Formulation in Physical Domain

In a dual assembly formulation, the full set of global DOF is retained, i.e., all interface DOF are present as many times as there are subdomains connected on the corresponding node. From Eq. (5), the dual assembled system is obtained by satisfying a priori the interface equilibrium. This is obtained by choosing the interface forces in the form

$$\mathbf{g} = -\mathbf{B}^T\boldsymbol{\lambda}$$

Here, $\boldsymbol{\lambda}$ are Lagrange multipliers, corresponding physically to the interface force intensities. By choosing the interface forces in this form, they act in opposite directions for any pair of dual interface degrees of freedom, due to the construction of Boolean matrix $\mathbf{B}$. The equilibrium condition is thus written

$$\mathbf{L}^T\mathbf{g} = -\mathbf{L}^T\mathbf{B}^T\boldsymbol{\lambda} = \mathbf{0}$$

Because it was shown that $\mathbf{L}^T$ was the nullspace of $\mathbf{B}^T$, see Eq. (7), this condition is always satisfied. Consequently, the system of Eq. (5) is now described by

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{f} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases}$$

In matrix notation, one finds the dually assembled system as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{u}} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{u}} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{9}$$

Clearly, $\boldsymbol{\lambda}$ are the Lagrange multipliers associated with the compatibility condition. Dual approaches were already considered in the early days of finite element theory, but dual assembly became really popular in the 1990s as a way to implement efficient solvers on parallel processing computers. This led to the family of parallel solvers known as dual Schur complement methods or FETI (finite elements tearing and interconnecting) [27].

B. Coupling in the Frequency Domain

When the dynamics of subsystems are obtained from measurements, one typically measures their frequency response for several inputs and outputs. (One could also consider time responses in time such as impulse responses. Nevertheless, this is usually not done in the context of dynamic substructuring.) In that case, the mass, damping, and stiffness properties of the system are not known separately like in the physical domain considered before. The equations of motion, as well as the coupling conditions, can be transformed from the physical domain into the frequency domain using the Fourier transform. To this end, the mechanical subsystems must be linear (or linearized), time invariant (i.e., constant parameters), and in steady state (i.e., transient effects must have damped out). Performing a Fourier transform on Eq. (5) then gives the following set of governing equations in the frequency domain:

$$\begin{cases} \mathbf{Z}(\omega)\mathbf{u}(j\omega) = \mathbf{f}(\omega) + \mathbf{g}(\omega) \\ \mathbf{B}\mathbf{u}(\omega) = \mathbf{0} \\ \mathbf{L}^T\mathbf{g}(\omega) = \mathbf{0} \end{cases} \tag{10}$$

Clearly, from the context of the equation, $\mathbf{u}$, $\mathbf{f}$, $\mathbf{g}$ represent the complex amplitude of the harmonic response and forces. Here, $\mathbf{Z}$ is the block-diagonal matrix containing the dynamic stiffness matrices of the substructures, i.e.,

$$\mathbf{Z}(j\omega) \triangleq -\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}$$

with j being the unit imaginary number. The same procedure as before can be followed to obtain either a primal or dually assembled system of equations.

1. Primal Formulation in the Frequency Domain

To obtain the primal system of equations, interface compatibility is again imposed by choosing a unique set of interface DOF as in Eq. (6). Analog to the physical domain coupling, one now finds

$$\tilde{\mathbf{Z}}\mathbf{q} = \tilde{\mathbf{f}} \tag{11}$$

where

$$\begin{cases} \tilde{\mathbf{Z}} \triangleq \mathbf{L}^T\mathbf{Z}\mathbf{L} \\ \tilde{\mathbf{f}} \triangleq \mathbf{L}^T\mathbf{f} \end{cases}$$

are the primally assembled frequency response function matrices and forcing amplitudes.

2. Dual Formulation in the Frequency Domain

To obtain a dual assembled system from Eq. (10), the equilibrium condition is imposed by choosing the interface forces as $\mathbf{g} = -\mathbf{B}^T\boldsymbol{\lambda}$. Analog to the dual coupling in the physical domain, one finds

$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{12}$$

Observe that, in experimental dynamics, one measures in free–free conditions, imposing forces on points and letting the remaining structure free, which results in the receptance matrix representation $\mathbf{Y}$ of the subsystems. Hence, one does not have direct access to the dynamic stiffness $\mathbf{Z}$ but to the information of its inverse. Therefore, the form written in Eq. (12) is not directly applicable. Eliminating the Lagrange multiplier (LM) in the system of Eq. (12), one finds the so-called dual interface problem in the frequency domain, suitable for the coupling of receptance matrices obtained with experimental data [28]:

$$\mathbf{u} = \mathbf{Y}\mathbf{f} - \mathbf{Y}\mathbf{B}^T(\mathbf{B}\mathbf{Y}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{Y}\mathbf{f} \tag{13}$$

where $\mathbf{Y} \triangleq \mathbf{Z}^{-1}$.

C. Coupling of Reduced Components

Substructures can be represented in an approximated manner by assuming that the degrees of freedom are in a reduced space of the full solution space of the structural component. This is often done in numerical simulation to reduce the computational cost, or in experimental substructuring when the behavior of the components is reconstructed from experimentally identified shapes (e.g., static responses and eigenmodes). Historically, the first methods published considered substructural representations including true eigenmodes of the components and were thus called "component-mode synthesis" [11]. Nowadays, the denomination component-mode synthesis is understood as "the construction of substructures based on a reduced space," the term "mode" being understood as any vector of the reduction space, such as a Ritz vector. The discussion in this section relates to the modal domain and component-mode synthesis mentioned in Fig. 2.

The substructure description may contain different kinds of information, but a dynamic description of the coupled system typically includes some sort of dynamic component modeshapes and/or a representation of the substructure's quasi-static response. So, in the modal domain "substructuring method" usually refers to a process where the substructure dynamics are approximated in a reduction basis.

Component-mode synthesis methods became particularly popular among the engineering community as a reduction method for finite element models. Usually, the CMS method comprises some kind of modal analysis on the substructure, of which the obtained modal vectors are used to reduce the equations of motion from the physical to the modal domain. By doing so, the full set of physical coordinates is reduced to a smaller set of *generalized* coordinates. (An important issue in the coupling of reduced components is the verification of the quality of the reduced models. To this end, one could, for instance, use an effective modal mass criterion [29]. However, a detailed discussion on this subject is out of the scope of this paper.) Note that if a full set of eigenmodes is taken, i.e., modal truncation is avoided, the reduction actually becomes a transformation to *modal* coordinates. The number of coordinates remains the same. The reduction is accomplished substructurewise by a reduction matrix $\mathbf{R}$, which is a block-diagonal matrix consisting of the substructures' individual reduction matrices:

$$\mathbf{u} \simeq \mathbf{R}\boldsymbol{\eta} \qquad \mathbf{R} \triangleq \mathrm{diag}(\mathbf{R}^{(1)}, \ldots, \mathbf{R}^{(n)}) \tag{14}$$

where $\boldsymbol{\eta}$ is a vector of generalized coordinates. As stated before, the substructure reduction matrix $\mathbf{R}$ can contain different kinds of dynamic component modeshapes. Substituting the approximation Eq. (14) into the equilibrium Eq. (1) of a substructure, one finds

$$\mathbf{M}^{(s)}\mathbf{R}\ddot{\boldsymbol{\eta}}^{(s)} + \mathbf{C}\mathbf{R}\dot{\boldsymbol{\eta}}^{(s)} + \mathbf{K}^{(s)}\mathbf{R}\boldsymbol{\eta}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} + \mathbf{r}^{(s)} \tag{15}$$

where the explicit time dependency is again omitted for clarity. In this equation, $\mathbf{r}^{(s)}$ is a residual force introduced due to the fact that the approximation given in the reduced basis cannot usually represent the exact solution. The reduced equilibrium equations are then obtained by imposing that the residual force must be zero in the reduction space, namely,[‡] $\mathbf{R}^{(s)^T}\mathbf{r}^{(s)} = \mathbf{0}$. One thus finds the reduced equations of motion of the decoupled subsystems as

$$\mathbf{M}_m\ddot{\boldsymbol{\eta}} + \mathbf{C}_m\dot{\boldsymbol{\eta}} + \mathbf{K}_m\boldsymbol{\eta} = \mathbf{f}_m + \mathbf{g}_m \tag{16}$$

The subscript m denotes the fact that the matrices are transformed to the modal domain:

$$\begin{cases} \mathbf{M}_m \triangleq \mathbf{R}^T\mathbf{M}\mathbf{R} \\ \mathbf{C}_m \triangleq \mathbf{R}^T\mathbf{C}\mathbf{R} \\ \mathbf{K}_m \triangleq \mathbf{R}^T\mathbf{K}\mathbf{R} \\ \mathbf{f}_m \triangleq \mathbf{R}^T\mathbf{f} \\ \mathbf{g}_m \triangleq \mathbf{R}^T\mathbf{g} \end{cases} \tag{17}$$

If the subsystems are coupled, both the compatibility and force equilibrium equations need to be imposed on the subsystems in generalized coordinates as well. The compatibility condition is transformed to generalized coordinates as

$$\mathbf{B}_m\boldsymbol{\eta} = \mathbf{0} \tag{18}$$

$$\mathbf{B}_m \triangleq \mathbf{B}\mathbf{R} \tag{19}$$

which is still a strong (or exact) compatibility requirement on the interface. A global set of generalized coordinates satisfying the compatibility condition can be found in a way analog to what was done to couple the physical subdomains:

$$\boldsymbol{\eta} = \mathbf{L}_m\boldsymbol{\xi} \tag{20}$$

Here $\boldsymbol{\xi}$ expresses the unique set of generalized coordinates of the assembled system and matrix $\mathbf{L}_m$ is the primal assembly operator (or, by comparison of the physical assembly, the localization matrix) associated with the generalized coordinates. Substituting Eq. (20) in the compatibility condition Eq. (18), one must have

$$\mathbf{B}_m\mathbf{L}_m\boldsymbol{\xi} = \mathbf{0} \quad \forall\ \boldsymbol{\xi} \tag{21}$$

meaning that $\mathbf{L}_m$ must span the nullspace of $\mathbf{B}_m$ if the interface is to be assembled in a fully compatible way:

$$\mathbf{L}_m = \mathrm{null}(\mathbf{B}_m) = \mathrm{null}(\mathbf{B}\mathbf{R})$$

The operator $\mathbf{L}_m$ is in general non-Boolean because $\mathbf{B}\mathbf{R}$ is in general non-Boolean. However, if the interface degrees of freedom of the substructures are kept as generalized DOF in the set $\boldsymbol{\eta}$, then $\mathbf{B}\mathbf{R}$, and thus $\mathbf{L}_m$, will still be Boolean.

By taking into account the coupling conditions in the transformed domain, one finds analog to Eq. (5) the following set of equations for the assembled structure:

$$\begin{cases} \mathbf{M}_m\ddot{\boldsymbol{\eta}} + \mathbf{C}_m\dot{\boldsymbol{\eta}} + \mathbf{K}_m\boldsymbol{\eta} = \mathbf{f}_m + \mathbf{g}_m \\ \mathbf{B}_m\boldsymbol{\eta} = \mathbf{0} \\ \mathbf{L}_m^T\mathbf{g}_m = \mathbf{0} \end{cases} \tag{22}$$

The model reduction weakens the force equilibrium condition as explained previously, i.e., the system response is computed only for forces that can be represented in the modal space. This compromise also holds for the interface forces. This can be seen by observing that,

[‡]Note that this principle is applied in a virtual work or variational principle approach as well.

recalling definition Eq. (17), one can also write

$$\mathbf{L}_m^T \mathbf{g}_m = \mathbf{L}_m^T \mathbf{R}^T \mathbf{g} = \mathbf{0} \tag{23}$$

Also, according to Eqs. (19) and (21), one finds $\mathbf{B}_m \mathbf{L}_m \boldsymbol{\xi} = \mathbf{B}(\mathbf{R}\mathbf{L}_m)\boldsymbol{\xi} = \mathbf{0}$ for all $\boldsymbol{\xi}$, meaning that $\mathbf{R}\mathbf{L}_m$ is the part of the nullspace of $\mathbf{B}$ that exists in the transformation space $\mathbf{R}$ or, in other words, $\mathbf{R}\mathbf{L}_m$ represents a subspace of $\mathbf{L}$. Hence, comparing Eq. (23) to Eq. (4), it can be understood as a weak equilibrium condition on the interface forces, exactly like the reduction process at the substructure level induces a weakening of the substructure equilibrium [see Eq. (15)].

Using the Eqs. (22) as a starting point, one can again assemble the substructures in a primal or dual manner, respectively.

1. Primal Formulation

In the primal formulation, direct use is made of the unique choice of interface DOF in global coordinates, i.e., Eq. (20). Substitution of the equation into Eq. (22) yields

$$\begin{cases} \mathbf{M}_m \mathbf{L}_m \ddot{\boldsymbol{\xi}} + \mathbf{C}_m \mathbf{L}_m \dot{\boldsymbol{\xi}} + \mathbf{K}_m \mathbf{L}_m \boldsymbol{\xi} = \mathbf{f}_m + \mathbf{g}_m \\ \mathbf{B}_m \mathbf{L}_m \boldsymbol{\xi} = \mathbf{0} \\ \mathbf{L}_m^T \mathbf{g}_m = \mathbf{0} \end{cases} \tag{24}$$

The second line in Eq. (24) is zero, as $\mathbf{L}_m = \text{null}(\mathbf{B}_m)$. Premultiplication of the remaining equations with $\mathbf{L}_m^T$ and noting that $\mathbf{L}_m^T \mathbf{g}_m$ must be equal to zero gives the primal system of equations of the coupled structure as

$$\tilde{\mathbf{M}}_m \ddot{\boldsymbol{\xi}} + \tilde{\mathbf{C}}_m \dot{\boldsymbol{\xi}} + \tilde{\mathbf{K}}_m \boldsymbol{\xi} = \tilde{\mathbf{f}}_m \tag{25}$$

with

$$\begin{cases} \tilde{\mathbf{M}}_m \triangleq \mathbf{L}_m^T \mathbf{M}_m \mathbf{L}_m \\ \tilde{\mathbf{C}}_m \triangleq \mathbf{L}_m^T \mathbf{C}_m \mathbf{L}_m \\ \tilde{\mathbf{K}}_m \triangleq \mathbf{L}_m^T \mathbf{K}_m \mathbf{L}_m \\ \tilde{\mathbf{f}}_m \triangleq \mathbf{L}_m^T \mathbf{f}_m \end{cases}$$

2. Dual Formulation

The dually assembled system is obtained when imposing the interface equilibrium by choosing the interface forces in generalized coordinates in the form:

$$\mathbf{g}_m = -\mathbf{B}_m \boldsymbol{\lambda}$$

where $\boldsymbol{\lambda}$ corresponds to the Lagrange multipliers associated to the interface intensities in generalized coordinates. The force equilibrium is now always satisfied, as

$$\mathbf{L}_m^T \mathbf{B}_m \boldsymbol{\lambda} = \mathbf{0}$$

The equations of motion of the dually assembled system in generalized coordinates can thus be written as

$$\begin{cases} \mathbf{M}_m \ddot{\boldsymbol{\eta}} + \mathbf{C}_m \dot{\boldsymbol{\eta}} + \mathbf{K}_m \boldsymbol{\eta} + \mathbf{B}_m^T \boldsymbol{\lambda} = \mathbf{f}_m \\ \mathbf{B}_m \boldsymbol{\eta} = \mathbf{0} \end{cases}$$

In matrix-vector form, the dually formulated system is equal to[§]

$$\begin{bmatrix} \mathbf{M}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\eta}} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_m & \mathbf{B}_m \\ \mathbf{B}_m^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{26}$$

Remark: Dual assembly and free-interface modes: It is interesting to observe that the dually assembled modally reduced problem Eq. (26) can also directly be obtained by considering the dually assembled physical problem Eq. (9) and reducing it, using the approximation

$$\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} \simeq \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{27}$$

where $\mathbf{I}$ is the identity matrix. Indeed, it can be easily verified that by substituting Eq. (27) into Eq. (9), one directly obtains Eq. (26).

In certain classes of component-mode synthesis techniques, the reduction space for the physical DOF includes so-called residual flexibility modes. Those modes correspond to the substructure's (quasi-) static response to a unit interface force and are typically used in combination with free-interface modes (for instance, in the MacNeal [13] and Rubin [12] methods).

In that case, two approaches can be considered. In the first approach, one can consider the reduction only on the DOF $\mathbf{u}$ and write

$$\mathbf{u} \simeq [\mathbf{R} \quad \mathbf{G}] \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{28}$$

and simply consider $\boldsymbol{\lambda}$ as a generalized DOF exactly as $\boldsymbol{\xi}$, and where $\mathbf{G}$ are the associated modes (typically residual flexibility modes). Using this reduction relation on the primally assembled physical problem Eq. (8), one finds the reduced systems as in Eq. (25) or in Eq. (26). This leads to the MacNeal [13] method as shown in [30] or to the so-called Craig–Chang method [31].

In a second approach, one can consider the reduction of the entire $(\mathbf{u}, \boldsymbol{\lambda})$ space by

$$\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} \simeq \begin{bmatrix} \mathbf{R} & \mathbf{G} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{29}$$

and apply this reduction directly on the dually assembled problem Eq. (9). It is important to observe that this does not yield the same reduced equations as in the first approach. Indeed, the reduced equations are now obtained by substituting Eq. (29) into Eq. (9) and premultiplying it by the transposition of the reduction matrix. This can schematically be written as

$$\begin{bmatrix} \mathbf{R}^T & \mathbf{0} \\ \mathbf{G}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \text{substructure equilibrium} \\ \text{interface compatibility} \end{bmatrix} \tag{30}$$

Clearly, whereas in the first approach the interface compatibility is exact and the interface equilibrium is approximate, here, the compatibility is now weakened, because the strong compatibility condition is now replaced by

$$\mathbf{G}^T(\text{substructure equilibrium}) + \text{interface compatibility}$$

indicating that one allows the interface to be slightly incompatible. This can be an efficient way to avoid interface locking when the displacement reduction space $\mathbf{R}$ is poor. The approach described by Eq. (30) leads, for instance, to the so-called dual Craig–Bampton method [32] or to the similar method proposed in [33].

IV. Frequency Response Function Based Substructuring and Component-Mode Synthesis Strategies

In the previous section, a general framework for the coupling of substructures was presented. This framework allows the classification of existing dynamic substructuring techniques in three main classes (see also Fig. 2): 1) coupling techniques in the physical domain, 2) coupling techniques in the frequency domain, and 3) coupling techniques in the modal domain.

As suggested in Sec. II, the process of coupling in the physical domain is equal to assembling the matrices of individual elements, as is done in the finite element method. However in the experimental community, coupling in the physical domain is very uncommon,

[§]Observe that mathematically speaking, the form obtained here is identical to the dual problem of the nonreduced problem except that now the compatibility matrix $\mathbf{B}_m$ is no longer Boolean.

because it is impossible to obtain the full experimental description of a subdomain in real structures.

Next, the most important dynamic substructuring methods found in the literature are classified according to the dynamic substructuring classes, frequency domain or modal domain.

A. Frequency-Based Substructuring

Basically, three fundamental frequency based substructuring methods can be found in the literature: 1) impedance coupling, 2) admittance coupling, and 3) SMURF/LM FBS coupling.

Classically, frequency domain substructuring was performed by primally coupling the dynamic stiffness. In the literature, this method commonly carries the name "impedance modeling/coupling" [17,20]. Because dynamic stiffness matrices are hard to find directly in practice, the data are obtained by inverting a measured structure's receptance matrix. Using the framework introduced in Sec. III, the impedance coupling method for n substructures with any number of arbitrary couplings is mathematically expressed by

$$\mathbf{q} = (\mathbf{L}^T\mathbf{Y}^{-1}\mathbf{L})^{-1}\mathbf{L}^T\mathbf{f}$$

Although this method is able to deal with measured FRFs directly, it is computationally inefficient and prone to severe error amplification. In the case of coupling two substructures, for example, it already involves three matrix inversions.[¶] In addition, the chance of round-off errors due to bad matrix conditioning is higher as well, as the matrices are inverted twice. If the calculations are not ill conditioned, their outcome will, however, be similar to the other methods because, theoretically, the method is exact and is equivalent to Eq. (5).

A more common substructuring technique in the frequency domain is the one proposed by Jetmundsen et al. in 1988 [19]. The method is sometimes suitably referred to in the literature as "admittance modeling" [34–36]. The method consists in coupling the receptance matrices of the substructures in a primal-like way, by partitioning the matrices according to the interface and internal degrees of freedom. In the case of coupling multiple arbitrary subsystems, use is made of the graph theory [19]. The method has an improved computational efficiency and better computational robustness against ill conditioning over the impedance coupling method. The original formulation of the method was generalized by Gordis et al. [37,38], and a similar method was proposed by Ren and Beards in 1993 [39].

In 1984, Crowley et al. [18] formulated the so-called SMURF method. This method uses the free-interface receptance matrices of the substructures to calculate the receptance matrix of the coupled system in a dual manner. However, as indicated earlier, this method gained little popularity as a dynamic substructuring method. This is probably due to the fact that it was initially intended mainly as a structural modification and troubleshooting tool [18]. In 2006, the method was reinvented and rewritten in a more straightforward manner to be used as a dynamic substructuring method, under the name Lagrange multiplier FBS (LM FBS) [28]. A more general framework for the method was presented in Sec. IV.B, which resulted in the final equation:

$$\mathbf{u} = \mathbf{Yf} - \mathbf{YB}^T(\mathbf{BYB}^T)^{-1}\mathbf{BYf}$$

or:

$$\mathbf{u} = \mathbf{Yf} - \mathbf{YB}^T\boldsymbol{\lambda} \tag{31}$$

$$\boldsymbol{\lambda} \triangleq (\mathbf{BYB}^T)^{-1}\mathbf{u}_\Delta \tag{32}$$

$$\mathbf{u}_\Delta \triangleq \mathbf{BYf} \tag{33}$$

[¶]One inversion of the assembled dynamic stiffness matrix and two inversions of the receptance matrices of each substructure in the block-diagonal matrix $\mathbf{Y}$ [17].

Equations (13) or (31–33) can be seen as a simpler formulation of the FBS method by Jetmundsen et al. [19] where a genuine dual assembly is used. The mechanical interpretation of Eqs. (31–33) will be explained briefly.

The responses of the individual subsystems upon the applied external excitation is equal to the first term $\mathbf{Yf}$ in Eq. (31). As a result, a gap $\mathbf{u}_\Delta$ is formed between the still-uncoupled subsystem interfaces [see Eq. (33)]. The interface forces of intensities $\boldsymbol{\lambda}$ applied on the common subsystem interfaces are then computed by Eq. (32) such that this gap is closed. Additional responses associated with the interface forces are expressed by $-\mathbf{YB}^T\boldsymbol{\lambda}$ in Eq. (31) and, as the gaps are now closed, Eq. (31) represents the response of the coupled subsystems.

Special attention to the determination of the interface force intensities $\boldsymbol{\lambda}$ is required. To this end, one needs to know the gap which the external forces applied on the individual subsystems introduced, e.g., $\mathbf{u}_\Delta$ from Eq. (33). Notice that the Boolean matrix arranges the subtraction of the subsystem interface displacements, e.g.,

$$\mathbf{u}_\Delta = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} Y^{(1)} & 0 \\ 0 & Y^{(2)} \end{bmatrix}\begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix} = Y^{(1)}f^{(1)} - Y^{(2)}f^{(2)} \tag{34}$$

where, for simplicity, $Y^{(1)}$ and $Y^{(2)}$ are the FRFs of two single DOF subsystems which are connected. Now the term $\mathbf{BYB}^T$ determines the gap, which results from an applied interface unit force on matching interface nodes. The flexibility of both subsystems are subtracted to this end by the construction of the Boolean matrix, i.e.,

$$\mathbf{BYB}^T\boldsymbol{\lambda} = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} Y^{(1)} & 0 \\ 0 & Y^{(2)} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\boldsymbol{\lambda} = (Y^{(1)} - Y^{(2)})\boldsymbol{\lambda} \tag{35}$$

Inversion of the interface flexibility matrix $\mathbf{BYB}^T$ gives a dynamic stiffness matrix describing the interface force needed to introduce a unit displacement gap at the interface DOF. Multiplication of this stiffness matrix with the gap, which was actually initiated by the external forces on the individual subsystems, gives the (negative) interface force that is needed to keep the subsystems together.

Compared with the Jetmundsen method [19], the LM FBS method shows its simplicity, as the subsystem receptance matrices do not need to be partitioned before coupling [28].

In the literature, numerous variations on these three methods have been proposed. Some important examples include the following:

1) Ferreira and Ewins proposed the addition of nonlinear joints [40,41].

2) Various authors [42–44] proposed numerical techniques to process the measured FRFs and thereby to improve the accuracy of the assembled systems in experimental substructuring, based on the Jetmundsen method [19]. Typically, a singular value decomposition is applied on measured FRF data to improve the numerical conditioning of the coupling calculations.

3) In the case of simple structures with only one coupling point to other structures, use can be made of the four-pole theory developed in the late 1950s [45]. As the method is restricted to single input and output data, this method can be seen as a restricted FBS method.

B. Component-Mode Synthesis

As the name tells, component-mode synthesis techniques use modes (in the general sense) to represent the dynamic behavior of a substructure. In experimental substructuring, CMS is possible in two different ways:

1) Direct identification of the matrices related to the reduced basis. The most basic approach is, for instance, to measure free–free eigenparameters of substructures: the eigenmodes represent the reduction basis; the modal mass, eigenfrequencies, and dampings then form the associated reduced matrices. The substructures can be assembled as explained in Sec. III.C. More accurate approaches based on this idea (including, for instance, residual and static modes) can be constructed [15], but applying more advanced identification of reduced dynamics requires precise and complex testing (see also Sec. V) and, to date, no clear success has been reported.

2) A different way to use the modal information obtained experimentally consists of synthesizing the admittance matrices of substructures by a spectral superposition. Obviously, residual terms of the modal identification can also be included. This approach can be seen as an FRF-based substructuring, as discussed in the previous section, where the modal identification step corresponds to a kind of filtration of the raw FRF measurements. (There is obviously a relation between the two methods discussed here, namely using the modes to synthesize the FRF and using them directly with the modal reduced matrices. If only modes are used, the two approaches are equivalent, but, when adding residual corrections, it is not clear if they lead to the same model [15].)

Component-mode synthesis started, in fact, as a reduction method, as outlined in the history discussion (Sec. II). Nearly all methods are based on the projection of the physical problem on a reduced basis consisting of general modes [see Eq. (16)] and they differ from one another by the nature of the modes used to approximate the physical space.

In the original method of Hurty in 1965 [9], the interface DOF are partitioned in a set that renders the substructures statically determined and in a set containing the remaining interface DOF. The modes used by Hurty for the reduction are then the rigid body modes associated with unit displacements of the statically determined DOF and the static response to unit displacements on the other interface DOF, the statically determined set being fixed. This method was slightly difficult to apply because it required partitioning of the interface sets. Therefore, the methods that became popular are the static condensation techniques of Guyan** [47] published around the same time as the Hurty method [9], and the Craig–Bampton method [11], which is still the most commonly used substructuring technique in structural dynamics.

In the Craig–Bampton method [11], not only static modes are used but also vibration modes associated with the dynamics of the DOF internal to the substructures, namely when the interface DOF are fixed. The reduced Craig–Bampton matrices are nearly diagonal and therefore lead to an efficient implementation in finite element codes.

Later, methods were proposed where the ingredients used to build the reduction basis consisted of so-called attachment modes, namely substructure responses corresponding to unit interface forces. In addition, free-interface modes where used to represent the internal dynamics of the substructures. This led to the methods of MacNeal [13], Rubin [12], and Craig and Chang [31].

Interesting reviews of component-mode synthesis methods can be found in [21,48,49]. Here, we will rapidly outline some of the important variations of the common and popular reduction methods:

1) Many authors have proposed different modes to build a reduction basis that better approximates the substructures' dynamics.

a) Already at the time when Hurty published the idea of substructuring, the branch method [10] was proposed in which the reduction modes are evaluated by taking into account the influence of neighboring substructures.

b) Several authors have proposed to replace the static modes in the reduction basis by quasi-static modes related to a dynamic stiffness matrix obtained by shifting around a central frequency [50–52].

c) Some authors have proposed to add masses to the interface when measuring or computing the substructure modes to account for the inertia loading of the neighboring substructures [53,54].

d) Instead of using vibration modes (which represent the general dynamics of the substructures) one can also use the so-called Krylov vectors related to the interface loading [55], which, in fact, arise from the concept of the load-dependent vectors proposed by Wilson et al. for model reduction [56].

e) A way to include both true eigenmodes and Krylov vectors consists of adding to the reduction basis so-called modal truncation augmentation vectors that represent a kind of mode acceleration correction. Examples of such higher order corrections can be found in [57–60].

2) Some rare publications have tackled the issue of reducing the interface problem. This is an important issue because in many practical applications the number of degrees of freedom on the interface is still unnecessarily high. A second reduction step for the interface DOF can then be performed and typically the reduction space is obtained from a preliminary Guyan reduction step [61–63].

3) Some authors have proposed to iteratively construct or improve the reduction basis, using the error residual from the previous iteration [61–66].

4) Multilevel substructuring consists of performing nested partitioning of the system and reducing the substructures successively starting at the lowest level. Any substructuring technique can be applied in a multilevel manner. One popular technique is the automatic multilevel substructuring method or AMLS [67,68]

5) The method proposed in [69], similar to the concept of statically determined DOF set by Hurty [9], introduces explicit rigid DOF at the global level to have direct access to the rigid body motion of the entire structure.

6) For damped systems, classical reduction basis can lead to full reduced damped matrices [70] or might not be adequate for an efficient reduction. In that case, some authors have proposed to consider the state-space form of the equation and use the associated modes and Krylov vectors [71–73].

7) Building substructured models that undergo parametric modifications (like in modal updating or optimization procedures) is an active research area. Several techniques to update or enrich the reduction basis to efficiently reduce families of models can be found in [74–77].

This overview and classification of component mode synthesis covers most of the important trends in the field, although the reference list mentioned here is by no means exhaustive. Note also that we have not touched the topics of substructuring for nonlinear and for multifield problems. These are currently active research fields but are beyond the scope of this paper.

V. Difficulties in Experimental Dynamic Substructuring

In experimental dynamic substructuring, some difficulties have to be dealt with to avoid an erroneous analysis. These difficulties all originate from the inability to properly measure all the subsystem's properties.

As will be seen, the FBS and CMS methods both have some distinct differences in the errors encountered. In general, one may say that the CMS method is more appropriate in case the (sub)structure is suitable for a modal identification. If the modal identification is performed well, good results can be obtained. However, in case a (sub)structure has high damping, special frequency dependencies (such as in rubber components) or a high modal density, direct use of the measured FRFs with the FBS method will probably yield better results.

Next, the main difficulties encountered are discussed, as well as solutions proposed in the literature.

A. Truncation Errors

A problem encountered in experimental substructuring with the CMS method is that of modal truncation. Modal truncation means that not all the modal degrees of freedom, describing the subsystem's dynamics, are contained in the subsystem's description. This is a common problem in experimental modal analysis (EMA), for which the concept of residual flexibility was developed [78,79]. If a subsystem is identified with an EMA and is afterward used in a DS calculation, the inclusion of these residual terms is essential. (The same holds for numerically obtained FRF data.) If the residual flexibility is not included, the substructure will behave more stiffly, as it has less degrees of freedom to deform in [80]. Shifts in resonance frequency of the total coupled system can then be expected. Note that

**Also known as the Guyan–Irons reduction, Irons having proposed the same approach later in his frontal solver [46].

residual flexibility is only an approximation of the higher modes, and so an error will still be made in the coupling of the subsystems.

A modal analysis on a subsystem might not always be possible, as the substructure might have too much damping, has frequency-dependent dynamic stiffnesses, or has a modal density that is too high. In such cases, direct coupling with the measured FRF data is the best option, in which the residual terms are included in the data naturally.

B. Rotational Degrees of Freedom

A lot has been written on the importance of rotational degrees of freedom in dynamic substructuring [81–83]. The measurement of these rotations, and the application of torques to excite them, is very difficult to accomplish in practice. However, if not taken into account, a large number of entries in the (receptance) matrix are omitted. The influence of omitting rotational information strongly depends on the component's interface flexibility [84].

Different approaches exist to tackle the rotational DOF issue. Basically, one can either put effort into measuring them [85] or expand translational data to reconstruct the information for the rotational DOF [86,87]. In particular, if one assumes that the interface has only local rigid motions, one can construct its response from a minimum of six coupling DOF at three nodes. This kind of procedure will only yield good results up to frequencies where local deformation between the interface nodes starts to take place.

C. Continuity of Interface

In all practical applications, the interface is in fact a continuous surface. Measurements, however, can usually only be performed on a small discrete number of points (unless field measurements such as holographic techniques are used). In that case, one must reconstruct the interface's continuous behavior using expansion strategies. The simplest method is to consider a rigid behavior around measured points: as discussed in the preceding point, one then describes the interface with rotational degrees of freedom. More complex interface deformation modes can be reconstructed if sufficient interface points are measured. Methods previously proposed in the field of model updating can be used, such as the system equivalent reduction expansion process (SEREP method) [86], where static deformations obtained from a finite element model in the vicinity of the interface are used. A local finite element model can also be used to determine local dynamic modes for the expansion of the measurements onto the interface [88]. Recently, the experimental community started using these kind of multiple point connections, enhancing the experimental coupling results significantly [35,36,89,90].

D. Rigid Body Modes

Essential to the successful coupling of substructures with the CMS method is the inclusion of the substructures' rigid body mode (RBM) information [91]. If this information is not contained, erroneous coupling results are calculated. This then affects the total frequency range, because the structure will always move in a combined motion of flexible and rigid modes. Even at higher frequencies, the rigid modes are still excited and essential.

In the FBS method, no explicit attention is needed for the coupling algorithm itself; the RBM information is included naturally. However, setting up the experiment such that the rigid body modes are included properly is still required. In practice, this means low-stiffness air springs or elastic bands should be used, separating the RBM frequencies from the first elastic modes well [92].

E. Dynamics of Joints

Dynamic substructuring methods are sensitive to the coupling mechanisms which take place at the subsystem interfaces. As for the coupling mechanisms between substructures, different approaches exist. Usually, the coupling between the subsystems is either modeled as exact, i.e., $\mathbf{Bu} = \mathbf{0}$, or with linear flexible joints [93].

In many engineering structures, however, people found nonlinear coupling mechanisms between parts modeled as substructures. This nonlinear behavior originates, for instance, from friction between the bolted parts. Efforts are being made to develop nonlinear models to account for such mechanisms [94]. The engineer should be aware of this kind of coupling behavior and decide whether it should be taken into account.

F. Time Delay

One specific application of the dynamic substructuring concept is the field of real-time dynamic substructuring. In real-time dynamic substructuring, a hybrid model of the complete system is created by combining an experimental substructure with a numerical model describing the remainder of the system. This technique is, among others, useful when dealing with nonlinear substructures because no explicit dynamic model of the nonlinear substructure has to be constructed. The nonlinear dynamic behavior can be taken into account experimentally by including the substructure in the real-time substructuring loop. However, a bottleneck in these real-time substructuring techniques is the time delay due to the inherent dynamics of the actuators used for the structural testing [95]. Because this time delay troubles the real-time DS simulations and can even cause instability during the experiments, this subject has received a lot of research attention over the last years [25,96].

G. Experimental Errors

In the case of dynamic substructuring using experimental data, measurement errors affect the response of the coupled system. Numerous errors can be made; in the literature, different kinds are addressed. First there is the problem of random measurement noise, which is inherent to performing measurements. Especially for lightly damped structures, the signal-to-noise ratio can become very small [35] between eigenfrequencies and at antiresonances. Measurement averaging is therefore seen as a requirement in most cases.

If the FRFs are polluted with random measurement noise and testing artifacts (e.g., collocation errors and added mass effects) the coupling results will be erroneous. This effect strongly depends on which FRFs the random errors are related. The interface flexibility matrix $\mathbf{BYB}^T$, for example, needs to be inverted in the admittance and LM FBS coupling. Because of the matrix operation, small measurement errors can be significantly amplified, resulting in large errors in the FRFs of the coupled system. To get a feeling of how sensitive the coupling is to the random measurement errors, one can monitor the conditioning of the assembled flexibility operator $\mathbf{BYB}^T$. If the conditioning number is high, the calculation becomes more sensitive to small inaccuracies. To improve the robustness of FBS, a lot of effort is spent on filtration techniques, using, for instance, singular value decomposition to make the inversion less sensitive to small perturbations on the matrix entries [34,43,44,97,98]. Furthermore, a method was recently developed that allows analyzing the effect of these random measurement errors on the accuracy of the FBS algorithm [99].

When applying the CMS method, use is made of a modal identification. Although the modal parameters are also affected by the random errors due to measurement noise and testing artifacts, better results can be accomplished with this kind of filtration. As a mathematical model is deducted, matrix conditioning is of less importance.

Research has shown that antiresonances are difficult to measure in practice [100]. Because antiresonances of substructures often determine, for a significant part, the FRF of the assembled system (for instance, for very flexible substructures [99,101]), special care must be taken to measure the FRFs accurately over the entire frequency range. Therefore, the following points must be kept in mind: 1) Sensor positioning and alignment. For instance, antiresonances can be very sensitive to the exact location of the excitation; 2) unmeasured side forces introduced by a shaker/stinger combination; 3) signal processing errors, like leakage and bias errors due to the limited frequency resolution; 4) added mass introduced by the measurement equipment; 5) local nonlinearities of the (sub) structure; 6) lightly damped substructures can be severely affected by

the damping of the suspension [92]. Also, the stiffness of the suspension can result in shifts of the lower eigenfrequencies.

VI. Conclusions

In this paper, the concept of substructuring has been placed in a historical context, explaining, for instance, its relation to domain decomposition and model reduction. A general framework has been outlined, starting from a general decomposed formulation and including dual and primal assembly of the substructures in the physical, frequency, and modal domains. In the light of that framework, the most important methods proposed over the last decades have been classified, both for the frequency-based substructuring used in experimental dynamics and for the component-mode synthesis techniques. The paper was concluded by a brief discussion of some of the important open issues that still render experimental substructuring difficult to apply in practice.

Appendix: Construction of Boolean Matrices

This appendix illustrates the construction of the Boolean matrices $\mathbf{B}$ and $\mathbf{L}$. To this end, the general system shown in Fig. A1 is considered: this figure schematically shows the coupling of two general substructures. Both substructures consist of three nodes; substructure A has 4 DOF, whereas substructure B holds 5 DOF.

In this example, nodes 2 and 3 of substructure A are coupled to nodes 5 and 6 of substructure B, respectively. And so, three compatibility conditions should be satisfied:

$$\begin{cases} u_{2x} = u_{5x} \\ u_{2y} = u_{5y} \\ u_{3x} = u_{6x} \end{cases} \tag{A1}$$

To express this condition as in Eq. (3), i.e., $\mathbf{Bu} = \mathbf{0}$, the signed Boolean matrix $\mathbf{B}$ must be constructed. The total vector of degrees of freedom $\mathbf{u}$ is

$$\mathbf{u} = [\, u_{1y} \quad u_{2x} \quad u_{2y} \quad u_{3x} \quad u_{4x} \quad u_{4y} \quad u_{5x} \quad u_{5y} \quad u_{6x} \,]^T$$

The signed Boolean matrix $\mathbf{B}$ is now found as

$$\mathbf{B} = \begin{array}{c} \begin{matrix} u_{1y} & u_{2x} & u_{2y} & u_{3x} & u_{4x} & u_{4y} & u_{5x} & u_{5y} & u_{6x} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Every coupling term or, equivalently, every compatibility condition, corresponds to a line in the Boolean matrix $\mathbf{B}$. Therefore, in the general case where the coupled substructures comprise n degrees of freedom of which m are coupled interface DOF, the matrix $\mathbf{B}$ has size $m \times n$. In this example, $n = 9$ and $m = 3$; the size of $\mathbf{B}$ is 3×9. It can easily be seen that the condition $\mathbf{Bu} = \mathbf{0}$ is equivalent to the three compatibility equations in Eq. (A1).

From this signed Boolean matrix, the Boolean localization matrix $\mathbf{L}$ is found by computing the nullspace. In this example, this gives

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The set of unique interface DOF that is chosen for this example is found as[††]

[††]The interface DOF of substructure B are retained.

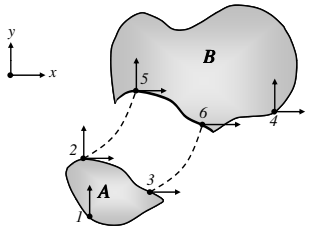

Fig. A1 Coupling of two general substructures.

$$\mathbf{q} = [\, u_{1y} \quad u_{4x} \quad u_{4y} \quad u_{5x} \quad u_{5y} \quad u_{6x} \,]^T$$

Indeed, the Boolean matrix $\mathbf{L}$ transforms this unique set of degrees of freedom to the total set of DOF:

$$\mathbf{u} = \mathbf{Lq} = \begin{bmatrix} u_{1y} \\ u_{5x} = u_{2x} \\ u_{5y} = u_{2y} \\ u_{6x} = u_{3x} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \end{bmatrix}$$

In addition, the Boolean localization matrix $\mathbf{L}$ describes the force equilibrium naturally as well:

$$\mathbf{L}^T\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g_{2x} \\ g_{2y} \\ g_{3x} \\ 0 \\ 0 \\ g_{5x} \\ g_{5y} \\ g_{6x} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_{2x} + g_{5x} \\ g_{2y} + g_{5y} \\ g_{3x} + g_{6x} \end{bmatrix} = \mathbf{0}$$

To satisfy the equilibrium condition, the connection forces on dual degrees of freedom must thus sum to zero.

Finally, it should be noted that there is another way to obtain the matrix $\mathbf{L}$ from $\mathbf{B}$. To this end, partitioning of the global set of DOF into sets of unique $\mathbf{u}_u$ and redundant $\mathbf{u}_r$ coordinates is required. The unique coordinates are all the internal DOF plus one set of interface DOF. The redundant coordinates are formed by the dual interface DOF. Partitioning Eq. (3) then gives

$$[\, \mathbf{B}_{rr} \quad \mathbf{B}_{ru} \,] \begin{bmatrix} \mathbf{u}_r \\ \mathbf{u}_u \end{bmatrix} = \mathbf{0}$$

Here, $\mathbf{B}_{rr}$ is a nonsingular square submatrix of $\mathbf{B}$. From this partitioned equation, it is clear that the redundant DOF can be found from the unique DOF as

$$\mathbf{u}_r = -\mathbf{B}_{rr}^{-1}\mathbf{B}_{uu}\mathbf{u}_u$$

Because the Boolean localization matrix $\mathbf{L}$ builds the set of global DOF from a set of unique DOF (i.e., $\mathbf{u} = \mathbf{Lq}$), one finds $\mathbf{L}$ directly from the partitioned compatibility equation:

$$\mathbf{u} = \mathbf{Lq} = \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} -\mathbf{B}_{rr}^{-1}\mathbf{B}_{ru} \\ \mathbf{I}_{uu} \end{bmatrix} \mathbf{u}_u \tag{A2}$$

In this example, this gives

$$\mathbf{B} = \begin{array}{c} \begin{matrix} u_{2x} & u_{2y} & u_{3x} & u_{1y} & u_{4x} & u_{4y} & u_{5x} & u_{5y} & u_{6x} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Computing the Boolean localization matrix from Eq. (A2) then gives

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It can easily be verified that this is indeed equal to the nullspace of the partitioned **B** matrix.

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